A remark on operator-norm convergence of Trotter-Kato product formula

著者	Tamura Hiroshi
journal or publication title	Integral Equations and Operator Theory
volume	37
number	3
page range	350-356
year	2000-01-01
URL	http://hdl.handle.net/2297/3991

A REMARK ON OPERATOR-NORM CONVERGENCE OF TROTTER-KATO PRODUCT FORMULA

Hiroshi TAMURA

An example is given which clarifies the present situation of the operator norm convergence of Trotter-Kato product formula. It shows that the rate of convergence of the formula with respect to the operator norm obtained in [NZ2] is best possible. It also yields a counter-example of the operator norm convergence of the formula in another case.

1 INTRODUCTION

Let A and B be nonnegative self-adjoint operators on a Hilbert space H with domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively. Then

$$\lim_{n\to\infty}\left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-t(A+B)} \tag{1}$$

holds in the sense of the strong convergence of operators, if $A+B$ is essentially self-adjoint on $\mathcal{D}(A)\cap\mathcal{D}(B)$. Moreover, if $\mathcal{D}(A^{1/2})\cap\mathcal{D}(B^{1/2})$ is dense in H, then $A+B$ is well-defined as the form-sum and (1) also holds in the strong convergence sense.

These are known as Trotter-Kato product formulas, (See, e.g., [T, C, K]) which have important applications in mathematical physics, e.g., through construction of Feynman path integral formulas. We refer to the former as Trotter's theorem and the latter as Kato's theorem.

In [R], Rogava proved that the formula (1) holds in the sense of operator-norm convergence if $\mathcal{D}(A)\subset\mathcal{D}(B)$ and $A+B$ is self-adjoint on $\mathcal{D}(A)$. After then, a series of works on this subject have appeared[DIT, IT1, IT2, NZ1, NZ2]. They extended the problem and improved the result of [R] in various senses. One of the typical results is the following.

THEOREM A [NZ2] *If the self-adjoint operators A and B satisfy, for some $\alpha\in(1/2,1)$ and $a\in(0,1)$,*

(i) $A\geq I, \qquad B\geq O\,;$

(ii) $\mathcal{D}(A^\alpha)\subset\mathcal{D}(B^\alpha), \quad ||B^\alpha x||\leq a\,||A^\alpha x||$ *for all* $x\in\mathcal{D}(A^\alpha)$*;*

(iii) $\mathcal{D}((A+B)^\alpha)\subset\mathcal{D}(A^\alpha)$,

then there is a constant $c>0$ such that

$$\left|\left|e^{-t(A+B)}-\left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right|\leq\frac{c}{n^{2\alpha-1}} \tag{2}$$

holds uniformly in $t\geq0$. Here $A+B$ is the form-sum of A and B.

The aim of the note is to clarify the accuracy of the above result by means of an example. Namely we have the following theorem.

THEOREM B *For each $\alpha \in (0,1)$ and $a \in (0,1)$, there exist operators A and B on a Hilbert space H such that they satisfy the conditions (i), (ii) and (iii) of Theorem A, and the bound*

$$\left|\left|e^{-t(A+B)} - \left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right| \geq \frac{C_t}{n^{2\alpha-1}} \quad \text{for } \alpha \in (1/2,1) \text{ and } n \text{ large,}$$

$$\liminf_{n\to\infty} \left|\left|e^{-t(A+B)} - \left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right| \geq D_t \qquad \text{for} \quad \alpha \in (0,1/2],$$

where C_t and D_t are positive and continuous in $t > 0$.

Remark 1: Theorem B implies that the bound (2) of the Theorem A is best possible in its power of n.
Remark 2: The case $\alpha = 1/2$ is an example for which Trotter-Kato product formula does not hold in the operator-norm convergence sense, while it holds in the strong convergence sense by Kato's theorem. In other words, the operator-norm convergence of the product formula can not be extended to the cases $\alpha \leq 1/2$ in this abstract setting, as conjectured in [NZ2].

2 EXAMPLE

Before dealing with a concrete example, let us recall some basic facts on operators acting in a Hilbert space direct sum. Let $\{ H_n \}_{n\in\mathbb{N}}$ be a countable family of Hilbert spaces over $\mathbb{R}$. The inner product of H_n is denoted by $(\ ,\ )_n$ for each $n \in \mathbb{N}$. The Hilbert space direct sum H of the Hilbert spaces $\{ H_n \}_{n\in\mathbb{N}}$ is defined by

$$H = \left\{ x = (x_n)_{n\in\mathbb{N}} \,\middle|\, x_n \in H_n \, (\forall n \in \mathbb{N}),\ \sum_{n=1}^{\infty}(x_n, x_n)_n < \infty \right\}$$

with its inner product

$$(x,y) = \sum_{n=1}^{\infty}(x_n, y_n)_n.$$

Let A_n be a bounded nonnegative self-adjoint operator on H_n for each $n \in \mathbb{N}$. From the operators $(A_n)_{n\in\mathbb{N}}$, we construct a nonnegative self-adjoint operator A in H with domain

$$\mathcal{D}(A) = \left\{ x = (x_n)_{n\in\mathbb{N}} \in H \,\middle|\, \sum_{n=1}^{\infty}(A_n x_n, A_n x_n)_n < \infty \right\},$$

$$(Ax)_n = A_n x_n \qquad \text{for} \qquad x = (x_n)_{n\in\mathbb{N}} \in \mathcal{D}(A).$$

Let B_n be another bounded nonnegative self-adjoint operator on H_n for each $n \in \mathbb{N}$, and B the operator constructed from $(B_n)_{n\in\mathbb{N}}$ as above. Then the sum $A+B$ is essentially self-adjoint on its domain $\mathcal{D}(A)\cap\mathcal{D}(B)$. Note that $\mathcal{D}(A)\cap\mathcal{D}(B)$ is dense in H. The closure of $(\,A+B\,,\,\mathcal{D}(A)\cap\mathcal{D}(B)\,)$ is equal to the operator constructed from $(A_n+B_n)_{n\in\mathbb{N}}$, which is also denoted by $A+B$. This is also equal to the form-sum of A and B. Now we consider the strongly continuous contraction self-adjoint semigroup $(e^{-tA_n})_{t\geq 0}$ on H_n for each n. Let T_t be the operator on H defined by

$$(T_t x)_n = e^{-tA_n} x_n \qquad \text{for } t \geq 0,\ \ x = (x_n)_{n\in\mathbb{N}} \in H.$$

Then $(T_t)_{t\geq 0}$ is equal to the strongly continuous contraction semigroup generated by $-A$, i.e., $T_t = e^{-tA}$. Similarly, A^α is equal to the operator constructed from $(A_n^\alpha)_{n\in\mathbb{N}}$ for $\alpha > 0$.

Now let us give a concrete example. Under the above notation, we set $H_n = \mathbb{R}^2$ and consider the two bounded nonnegative self-adjoint operators

$$A_n = \lambda(E + n(S+E)), \qquad B_n = n(S\cos\theta_n + T\sin\theta_n + E)$$

on H_n for each $n \in \mathbb{N}$. Here the operators S, T and E are defined by the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$, we construct operators A and B respectively, as above. We also get $e^{-t(A+B)}$ and so on.

We choose parameters λ, $\theta_n \in (0, \pi/2]$ by setting

$$\cos\theta_n = 1 - \epsilon_n, \quad \epsilon_n = 2(2n)^{-\max\{1,2\alpha\}}, \quad \lambda = (2/a^2)^{1/2\alpha} > 1 \quad \text{for } \alpha, a \in (0,1).$$

Note that A and B depend on α and a. These operators have the following properties.

PROPOSITION 1 *The operators A and B, defined above for each α, $a \in (0,1)$, satisfy*

(i) $A \geq I$, $\qquad B \geq O$;
(ii) $\mathcal{D}(A^\alpha) \subset \mathcal{D}(B^\alpha)$, $\quad ||B^\alpha x|| \leq a\,||A^\alpha x||$ for all $x \in \mathcal{D}(A^\alpha)$;
(iii) $\mathcal{D}((A+B)^\alpha) \subset \mathcal{D}(A^\alpha)$.

Proof: Using the matrix

$$P_n = \begin{pmatrix} \cos(\theta_n/2) & \sin(\theta_n/2) \\ -\sin(\theta_n/2) & \cos(\theta_n/2) \end{pmatrix},$$

we have

$$B_n = P_n^T \begin{pmatrix} 2n & 0 \\ 0 & 0 \end{pmatrix} P_n, \quad A_n = \lambda \begin{pmatrix} 1+2n & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus (i) is obvious and we get (ii) because the bound

$$||B_n^\alpha x_n||_n^2 = (x_n, B_n^{2\alpha} x_n)_n = (2n)^{2\alpha}\Big(x_n^1 \cos(\theta_n/2) + x_n^2 \sin(\theta_n/2)\Big)^2$$

$$\leq 2(2n)^{2\alpha}(x_n^1)^2 + \epsilon_n (2n)^{2\alpha}(x_n^2)^2 \leq a^2\,||A_n^\alpha x_n||_n^2$$

holds for $x_n = (x_n^1, x_n^2)^T \in \mathbb{R}^2$.

For (iii), we use the matrix

$$Q_n = \begin{pmatrix} \cos(\varphi_n/2) & \sin(\varphi_n/2) \\ -\sin(\varphi_n/2) & \cos(\varphi_n/2) \end{pmatrix},$$

where $\varphi_n \in (0, \pi/2)$ is defined by

$$\tan\varphi_n = \frac{\sin\theta_n}{\lambda + \cos\theta_n}.$$

Then we have

$$A_n + B_n = (\lambda + n + n\lambda)E + n\sqrt{\lambda^2 + 2\lambda\cos\theta_n + 1}(S\cos\varphi_n + T\sin\varphi_n) \tag{3}$$

and

$$Q_n(A_n+B_n)^{2\alpha}Q_n^T$$
$$= \begin{pmatrix} (\lambda+n+\lambda n+n\sqrt{\lambda^2+2\lambda\cos\theta_n+1})^{2\alpha} & 0 \\ 0 & (\lambda+n+\lambda n-n\sqrt{\lambda^2+2\lambda\cos\theta_n+1})^{2\alpha} \end{pmatrix}.$$

So we get (iii) because of the bound

$$||A_n^\alpha x_n||_n^2 = (y_n, Q_n A_n^{2\alpha} Q_n^T y_n)_n$$
$$\leq 2\lambda^{2\alpha}\big((1+2n)^{2\alpha}\cos^2(\varphi_n/2) + \sin^2(\varphi_n/2)\big)(y_n^1)^2$$
$$+2\lambda^{2\alpha}\Big(\cos^2(\varphi_n/2) + (1+2n)^{2\alpha}\sin^2(\varphi_n/2)\Big)(y_n^2)^2$$
$$\leq C\,(y_n, Q_n(A_n+B_n)^{2\alpha}Q_n^T y_n)_n = C\,||(A_n+B_n)^\alpha x_n||_n^2,$$

where $y_n = (y_n^1, y_n^2)^T = Q_n x_n$ and C is a positive constant independent of n. □

PROPOSITION 2 *For each $\alpha \in (1/2, 1]$ and $a \in (0,1)$, there exists $n_0 \in \mathbb{N}$ which depends only on λ such that the bound*

$$\left|\left|e^{-t(A+B)} - \left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right| \geq C_t\, n\epsilon_n$$

holds for all $t > 0$ and $n > n_0$. For each $\alpha \in (0, 1/2)$ and $a \in (0,1)$, the bound

$$\liminf_{n\to\infty}\left|\left|e^{-t(A+B)} - \left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right| \geq D_t$$

holds for all $t > 0$. Here C_t and D_t are positive continuous functions of $t > 0$.

Proof: Note that the inequalities

$$\left|\left|e^{-t(A+B)} - \left(e^{-tA/n}e^{-tB/n}\right)^n\right|\right| \geq \left|\left|e^{-t(A_n+B_n)} - \left(e^{-tA_n/n}e^{-tB_n/n}\right)^n\right|\right|_n$$
$$\geq \frac{1}{2}\left|\mathrm{Tr}\, e^{-t(A_n+B_n)} - \mathrm{Tr}\left(e^{-tA_n/n}e^{-tB_n/n}\right)^n\right| \tag{4}$$

hold, where the norm in the first member means the operator norm of bounded operators on H, that in the second member is the operator norm on $H_n = \mathbb{R}^2$ and Tr in the third member is the trace of 2×2 matrices.

For later use, let us prepare the following formulas:

$$e^{-s\Lambda} = E\cosh s - \Lambda\sinh s, \qquad \mathrm{Tr}\, e^{-s\Lambda} = 2\cosh s, \tag{5}$$

where $\Lambda = S\cos\theta + T\sin\theta$.

To get the first formula, we expand the exponential and use $\Lambda^2 = E$, which is an immediate consequence of the identities

$$ST + TS = O, \qquad S^2 = T^2 = E.$$

The second formula follows from the first one and $\mathrm{Tr}\ \Lambda = 0$.

Thanks to (3) and the above formulas, we get

$$\mathrm{Tr}\ e^{-t(A_n+B_n)} = 2e^{-t(\lambda+n+n\lambda)} \cosh\left(tn\sqrt{\lambda^2+1+2\lambda\cos\theta_n}\right)$$

$$= 2e^{-t(\lambda+n+n\lambda)} \cosh\left(tn(\lambda+1) - \frac{t\lambda}{\lambda+1} n\epsilon_n + O(tn\epsilon_n^2)\right) \tag{6}$$

for n large. On the other hand, calculations using cyclic property of the trace and (5) lead

$$\mathrm{Tr}\left(e^{-tA_n/n}e^{-tB_n/n}\right)^n = \mathrm{Tr}\left(e^{-tA_n/2n}e^{-tB_n/n}e^{-tA_n/2n}\right)^n = e^{-t(\lambda+n+n\lambda)}$$

$$\times \mathrm{Tr}\left(e^{-t\lambda S/2}e^{-t(S\cos\theta_n+T\sin\theta_n)}e^{-t\lambda S/2}\right)^n = e^{-t(\lambda+n+n\lambda)}\mathrm{Tr}\left(a_nE - b_nS - c_nT\right)^n,$$

where a_n, b_n and c_n are real positive numbers defined by

$$a_n = \cosh t\lambda\cosh t + \sinh t\lambda \sinh t\cos\theta_n = \cosh t(\lambda+1) - \epsilon_n \sinh t\lambda \sinh t, \tag{7}$$

$$b_n = \sinh t\lambda \cosh t + \cosh t\lambda \sinh t \cos\theta_n,$$

$$c_n = \sinh t \sin\theta_n.$$

Since the identity

$$a_n^2 - b_n^2 - c_n^2 = 1$$

holds, there exist real positive numbers K_n and Θ_n which satisfy

$$a_n = \cosh K_n, \quad b_n = \sinh K_n \cos\Theta_n, \quad c_n = \sinh K_n \sin\Theta_n. \tag{8}$$

From (7) and (8), we get

$$K_n = t(\lambda+1) - \frac{\sinh t\lambda \sinh t}{\sinh t(\lambda+1)}\epsilon_n + O(t\epsilon_n^2).$$

Setting $s = K_n, \theta = \Theta_n$ in (5), we have

$$\mathrm{Tr}\left(e^{-tA_n/n}e^{-tB_n/n}\right)^n = e^{-t(\lambda+n+n\lambda)}\mathrm{Tr}\left(e^{-K_n(S\cos\Theta_n+T\sin\Theta_n)}\right)^n$$

$$= 2e^{-t(\lambda+n+n\lambda)}\cosh nK_n.$$

Recall $n\epsilon_n = 1$, when $\alpha \le 1/2$. Then this case of the proposition follows with

$$D_t = \frac{e^{-t\lambda}}{2}\left(\exp\left(-\frac{\sinh t\lambda \sinh t}{\sinh t(\lambda+1)}\right) - \exp\left(-\frac{t\lambda}{\lambda+1}\right)\right),$$

which is positive because of the following inequality for positive a and b

$$\frac{\sinh(a+b)}{a+b} - \frac{\sinh a}{a}\frac{\sinh b}{b} = \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\left(e^{ax}-e^{ay}\right)\left(e^{bx}-e^{by}\right)dx\,dy > 0.$$

In order to consider the $\alpha > 1/2$ case, let us introduce

$$\eta_n = \lambda + 1 - \sqrt{\lambda^2 + 2\lambda\cos\theta_n + 1} = \lambda + 1 - \sqrt{(\lambda+1)^2 - 2\lambda\epsilon_n}, \quad \epsilon_n = \frac{\lambda+1}{\lambda}\eta_n - \frac{1}{2\lambda}\eta_n^2.$$

Using Cauchy's mean value theorem, (7), (8) and the inequality $\sinh nx/\sinh x \geq e^{(n-1)x}$, we have

$$\cosh nK_n - \cosh nt(\lambda+1-\eta_n) = n\frac{\sinh nL_n}{\sinh L_n}\big(\cosh K_n - \cosh t(\lambda+1-\eta_n)\big)$$

$$\geq e^{(n-1)L_n} n\left(\left(1 - \frac{\epsilon_n}{2}\right)\cosh t(\lambda+1) + \frac{\epsilon_n}{2}\cosh t(\lambda-1) - \cosh t(\lambda+1-\eta_n)\right)$$

$$= e^{t(n-1)(\lambda+1+O(\epsilon_n))}\int_0^t n(t-s)\left(\left(1-\frac{\epsilon_n}{2}\right)(\lambda+1)^2\cosh s(\lambda+1)\right.$$

$$\left. + \frac{\epsilon_n}{2}(\lambda-1)^2\cosh s(\lambda-1) - (\lambda+1-\eta_n)^2\cosh s(\lambda+1-\eta_n)\right) ds$$

$$\geq e^{t(n-1)(\lambda+1+O(\epsilon_n))}\int_0^t (t-s)\big(\cosh s(\lambda+1) - \cosh s(\lambda-1)\big)\, ds$$

$$\times\Big(\frac{(\lambda+1)(3\lambda-1)}{2\lambda}n\eta_n - \frac{3\lambda^2+6\lambda-1}{4\lambda}n\eta_n^2 + \frac{n\eta_n^3}{2}\Big),$$

where $L_n = t(\lambda+1+O(\epsilon_n))$ is the mean value of K_n and $t(\lambda+1-\eta_n)$. In the last step, we have used the Jensen's inequality

$$\cosh\left(\left(1-\frac{\eta_n}{2}\right)s(\lambda+1) + \frac{\eta_n}{2}s(\lambda-1)\right) \leq \left(1-\frac{\eta_n}{2}\right)\cosh s(\lambda+1) + \frac{\eta_n}{2}\cosh s(\lambda-1).$$

Now the $\alpha > 1/2$ case of the proposition is obvious. □

Thus we have proved Theorem B.

In connection with Rogava's theorem, we note here the following property of our example. It implies that the condition of Trotter's theorem is not sufficient for the norm-convergence.

PROPOSITION 3 *The operators A and B, constructed above, satisfy*

$\mathcal{D}(A) \not\subset \mathcal{D}(B)$, $\mathcal{D}(B) \not\subset \mathcal{D}(A)$ and $A + B$ is essentially self-adjoint but not self-adjoint on $\mathcal{D}(A)\cap\mathcal{D}(B)$ for each $\alpha, a \in (0,1)$.

Proof: Put $x_n = (0, n^{-1+\alpha/2})^T \in \mathbb{R}^2$, then we have

$$\sum_{n=1}^{\infty}(x_n, x_n)_n < \infty, \quad \sum_{n=1}^{\infty}(A_n x_n, A_n x_n)_n < \infty, \quad \sum_{n=1}^{\infty}(B_n x_n, B_n x_n)_n = \infty,$$

i.e.,

$$x = (x_n)_{n\in\mathbb{N}} \in \mathcal{D}(A) - \mathcal{D}(B).$$

Similarly,

$$y = (y_n)_{n\in\mathbb{N}} \in \mathcal{D}(B) - \mathcal{D}(A)$$

and

$$z = (z_n)_{n\in\mathbb{N}} \in \mathcal{D}(A+B) - \mathcal{D}(A) \subset \mathcal{D}(A+B) - \big(\mathcal{D}(A)\cap\mathcal{D}(B)\big)$$

hold, where $y_n = n^{-1+\alpha/2}(-\sin(\theta_n/2), \cos(\theta_n/2))^T$, $z_n = n^{-1+\alpha/2}(-\sin(\varphi_n/2), \cos(\varphi_n/2))^T$ and $\mathcal{D}(A+B)$ denotes the domain of the form-sum $A+B$. $\square$

ACKNOWLEDGEMENTS

The author thanks Prof. Takashi Ichinose and Prof. Asao Arai for directing his attention to this subject. He also thanks Prof. Hideo Tamura for some comments.

REFERENCES

[C] Paul. R. Chernoff: Note on product formulas for operator semigroups, *J. Funct. Anal.* **2** (1968), 238–242.

[DIT] Atsushi Doumeki, Takashi Ichinose and Hideo Tamura: Error bounds on exponential product formulas for Schrödinger operators, *J. Math. Soc. Japan* **50** (1998), 359–377.

[IT1] Takashi Ichinose and Hideo Tamura: Error estimate in operator norm for Trotter-Kato product formula, *Integr. Equ. Oper. Theory* **27**(1997), 195–207.

[IT2] Takashi Ichinose and Hideo Tamura: Error estimate in operator norm of exponential product formula for propagators of parabolic equations, *Osaka J. Math.* **35** (1998), 751–770

[K] Tosio Kato: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups, *Topics in Funct. Anal., Ad. Math. Suppl. Studies* Vol. 3, 185–195 (I. Gohberg and M. Kac eds.), Acad. Press, New York 1978.

[NZ1] Hagen Neidhardt and Valentin A. Zagrebnov: On error estimates for the Trotter-Kato product formula, *Lett. Math. Phys.* **44**(1998), 169–186.

[NZ2] Hagen Neidhardt and Valentin A. Zagrebnov: Fractional powers of self-adjoint operators and Trotter-Kato product formula, to appear in *Integr. Equ. Oper. Theory*.

[R] Dzh. L. Rogava: Error bounds for Trotter-type formulas for self-adjoint operators, *Funct. Anal. Prilozhen* **27** (1993), 84–86: English transl. in *Funkt. Anal. Appl.* **27**(1993), 217–219.

[T] Hale F. Trotter: On the product of semi-groups of operators, *Proc. Amer. Math. Soc.* **10** (1959), 545–551.

Hiroshi TAMURA
Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa 920-1192
Japan
E-mail: tamura@@kappa.s.kanazawa-u.ac.jp